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A generic approach to boundary reflection in periodic media

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Abstract. Waves in periodic media, whose propagation is governed by nearest neighbour interaction, are investigated. The reflection and transmission coefficients are derived for a plane wave incident from medium 1 upon medium 2, without invoking common approximations. The derivation is valid for a class of waves including magneto- and electro-inductive waves, waves on short loaded dipoles, nanoparticles, coupled waveguides and acoustic waves in monatomic media. For this last case hitherto unknown microscopic reflection and transmission coefficients are derived and shown to reduce in the continuous limit to the well-known expressions in terms of acoustic impedances.

PACS. 41.20.Jb Electromagnetic wave propagation; radiowave propagation

1 Introduction

Waves in periodic media have been studied for well over a century, starting with acoustic waves, followed by X-ray diffraction, electron diffraction, the band structure of solids, and waves along a periodic set of electric four-pole circuits, to mention a few. The best book on the subject is probably that of Brillouin, published nearly 60 years ago [1], although there are good accounts in many books on electromagnetic theory, e.g. in that of Collin [2]. A plane electromagnetic wave incident upon a periodic medium poses a difficult problem. It needs sophisticated mathematics to find the reflection and transmission coefficients and all the excited higher order modes. For a recent study see Belov et al. [3]. If the complications caused by the boundary can be disregarded then the solution can be expressed in much simpler form as presented in Tretyakov's book [4].

A logical extension of these studies is to the case when a plane wave is incident from one periodic medium upon another one. Generally, solutions for reflection and transmission coefficients are obtained by making approximations. However, these are unnecessary, and exact solutions were obtained by Syms [5] nearly 20 years ago for waves on coupled waveguides and recently by Syms et al. [6] for magnetoinductive waves introduced earlier by Shamonina et al. [7,8].

The aim of this paper is to generalise the method in [5] to solve the boundary problem for all periodic media in which wave propagation can be attributed to nearest neighbour interaction. The basic relations for a single medium, namely the recursive equation, the wave assumption and the dispersion equation, are introduced in Section 2. The equations relating the elements across the boundary to each other, and their solution for the reflection and transmission coefficients, are presented in Section 3. The continuous limit is discussed in Section 4, the power relations in Section 5, and examples are given in Section 6. The special case of acoustic waves is treated in more detail in Section 7. Further generalisations are discussed in Section 8 and conclusions drawn in Section 9.

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2 Dispersion equation in a single medium

The general arrangement is shown in Figure 1. It is assumed that the temporal variation is in the form $\exp(j\omega t)$, where ω is the frequency and t is time. It is also assumed that the same type of waves propagate both in Medium 1 and in Medium 2, that their period is the same in the direction parallel to the boundary and the incident plane wave is perpendicular to the boundary. Let us take the distance between the elements in the periodic medium as d, and let y_n be some property of the nth element. Nearest neighbour interaction implies that the element n is coupled in some way to elements $n-1$ and $n+1$. Let this coupling be expressed by the recursive equation

$$gy_n + h(y_{n-1} + y_{n+1}) = 0. \qquad (1)$$

Here g and h are independent of space but may vary, for example, with frequency. Next, we shall assume that a lossless wave propagates in Medium 1. The wave may be written as

$$y_n = y_{oo} \exp(-jknd). \qquad (2)$$

Here y_{oo} is a constant and k is the propagation coefficient. Substituting equation (2) into equation (1) we obtain the

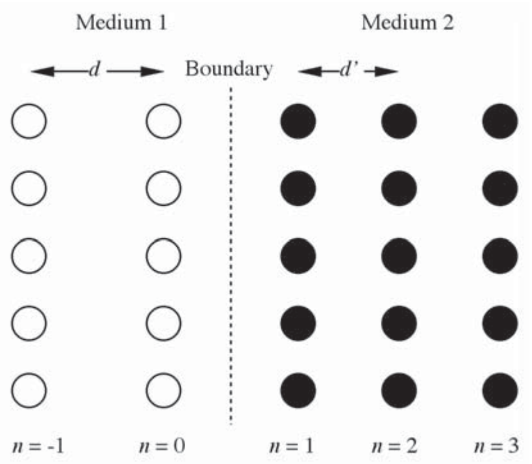


Fig. 1. General geometry for boundary matching between two periodic media.

dispersion equation in the form:

$$g + 2h\cos(kd) = 0. \quad (3)$$

The simple equations (1–3) form the basis for analysing a wide variety of wave propagation problems, as we now show.

3 Reflection and transmission coefficients

In Medium 2 we shall assume that the coefficients of the recursive equation are g' and h'. Having described the behaviour of the elements inside Media 1 and 2, next we look at the boundary. As shown in Figure 1, the columns nearest to the boundary are denoted '0' in Medium 1 and '1' in Medium 2. For the elements in these columns, the symmetry is broken. Crucially, the coupling across the boundary is not the same as the coupling towards the interior of the medium. Hence the nearest neighbour interactions across the boundary are described by the equations

$$\begin{aligned} gy_o + hy_{-1} + h_b y_1 &= 0 \\ g'y_1 + h_b y_o + h'y_2 &= 0. \end{aligned} \quad (4)$$

Here h_b is the coupling constant across the boundary. Equations (4) may be solved exactly, without the approximations common in almost all earlier works. For perpendicular incidence, the reflected and transmitted waves may be taken as:

$$\begin{aligned} y_n &= y_o\{\exp(-jnkd) + R\exp(jnkd)\} \quad \text{in Medium 1} \\ y_n &= y_o T\exp(-jn[kd]') \quad \text{in Medium 2.} \end{aligned} \quad (5)$$

Here $[kd]'$ is the phase change per element in Medium 2, and R and T are the reflection and transmission coefficients, respectively.

The two unknowns R and T may be found from equations (4) and (5). Using the dispersion equations to eliminate g and g' we obtain after a moderate amount of algebra:

$$\begin{aligned} R &= \{h_b^2\exp(-j[kd]') - hh'\exp(-jkd)\}/\{hh'exp(jkd) \\ &\quad - h_b^2\exp(-j[kd]')\} \\ T &= 2jhh_b\sin(kd)/\{hh'\exp(jkd) - h_b^2\exp(-j[kd]')\}. \end{aligned} \quad (6)$$

Equations (6) are more complicated than most commonly presented similar expressions. However, they can be simplified in the case when $h = h' = h_b$, which implies that the coupling coefficients are identical in the two media and across the boundary as well. This situation can arise only when the sole difference between the two media is that $g \neq g'$. Consequently, the values of k and k' calculated from the dispersion equations, will be different and equations (6) reduce to the simple form

$$\begin{aligned} R &= \{\exp(-j[kd]') - \exp(-jkd)\}/\{\exp(jkd) - \exp(-j[kd]')\} \\ T &= 2j\sin(kd)/\{\exp(jkd) - \exp(-j[kd]')\}. \end{aligned} \quad (7)$$

Equations (7) are immediately recognisable as more 'classical' coefficients. We also note that an expression identical to equation (7) appears in Tretyakov's book [4], but the underlying physics is quite different in the two cases. Here, we are concerned with the reflection and transmission of waves propagating in different periodic media, whereas Tretyakov's expression is valid when a plane electromagnetic wave is incident upon a periodic medium and higher order modes at the boundary can be disregarded. It is a coincidence that the same result is obtained.

4 The continuous limit

In the limit when the number of elements is very large and the wavelength is much longer than the distance between the elements, $kd \ll 1$ and the difference equation (1) can be reduced to the differential equation

$$gy + h\{d^2y/dx^2 - 2y\} = 0 \quad (8)$$

where x is a continuous distance along the array. The dispersion equation can be obtained by solving the above differential equation or, alternatively, by expanding equation (3) assuming that $kd \ll 1$. In either case we obtain

$$(kd)^2 = 2 + g/h. \quad (9)$$

It is instructive to look now at equations (7), the simplified form of the reflection and transmission coefficients, in the continuous limit. They reduce to

$$\begin{aligned} R &= (k' - k)/(k' + k) \\ T &= 2k/(k' + k). \end{aligned} \quad (10)$$

Equations (10) are quite familiar expressions, occurring for example when Schrödinger's equation is solved for an electron wave incident upon a potential barrier (see e.g. [9]). There are no periodic media in that case, simply a wave incident from one medium upon another one.

5 Power relations

Having found the reflection and transmission coefficients we still need to define the power in the wave. Returning to equation (1), we note that the constant terms g and h in equation (1) are generally either both real or both imaginary (see later examples). Thus, we may replace them with their moduli without loss of generality. Multiplying by y_n^*, we then obtain:

$$|g|y_n y_n^* + |h|(y_{n+1}y_n^* + y_{n+1}y_n^*) = 0. \qquad (11)$$

Forming a similar equation from the complex conjugate of equation (11), and then subtracting, we obtain:

$$|h|\{(y_{n+1}^* y_n - y_{n+1}y_n^*) - (y_n^* y_{n-1} - y_n y_{n-1}^*)\} = 0. \quad (12)$$

This result implies that the term $|h|(y_{n+1}^* y_n - y_{n+1}y_n^*) = |h|\ \mathrm{Im}\ (y_{n+1}^* y_n)$ is invariant with respect to n, and hence equation (12) represents a form of power conservation in a lossless system. In fact, the quantity

$$P = C|h|\ \mathrm{Im}\ (y_{n+1}^* y_n) \qquad (13)$$

can represent local power flow, if the constant C is determined from the product of group velocity and stored energy, as we have shown elsewhere for magneto-inductive waves [6]. The presence of the term $|h|$ serves to distinguish different media, and is important in boundary problems, as we now show.

If equation (13) represents power flow, then it should be conserved across a boundary, i.e. we should find the same value both in Medium 1 and in Medium 2. The variable y_n in Medium 1 is given by the upper equation (5). Substituting it into equation (13) gives the power in the form

$$P = C|y_o|^2|h|(1 - |R|^2)\sin(kd). \qquad (14)$$

In Medium 2, y_n is given by the lower equation (5). Substituting it into equation (13) we obtain similarly the power in Medium 2 as

$$P' = C|y_o|^2|h'||T|^2\sin[(kd)']. \qquad (15)$$

Substituting R and T from equations (6) into equations (14) and (15) we can show that $P = P'$, i.e. power is conserved across the boundary, as it should be.

6 Examples

As mentioned in the Introduction, the relationships derived are applicable to a fair number of different waves. We shall give below several examples. We begin with magnetoinductive waves, for which reflection and transmission coefficients have already been derived [6] for the two-dimensional case.

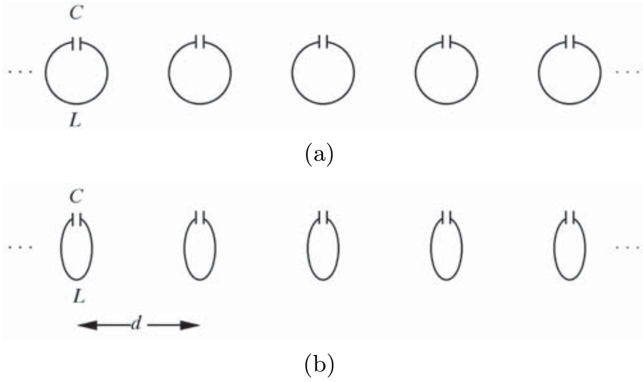


Fig. 2. Magneto-inductive waveguides in (a) axial and (b) planar configurations.

6.1 Planar and axial configurations of magnetoinductive waves

Figures 2a and 2b show one-dimensional arrays of capacitively loaded loops in the planar and axial configurations respectively. The variable is now the current flowing in the nth element, which is coupled to the currents in the $n-1$th and $n+1$th element. It follows from Kirchhoff's Law that

$$Z_o I_n + j\omega M(I_{n-1} + I_{n+1}) = 0. \qquad (16)$$

Here Z_o is the self-impedance and M is the mutual inductance between the loops. The difference between the axial and planar configurations is the sign of the mutual inductance: it is positive for the axial and negative for the planar case.

At a given frequency we may now obtain the reflection and transmission coefficients by substituting into equations (6)

$$h = j\omega M, \quad h' = j\omega M' \text{ and } \quad h_b = j\omega M_b \qquad (17)$$

where M, M' and M_b are the mutual inductances in Medium 1, Medium 2 and across the boundary respectively. The phase changes in the exponential terms, kd and $(kd)'$, may be obtained from the dispersion equations for a given ω, as noted above.

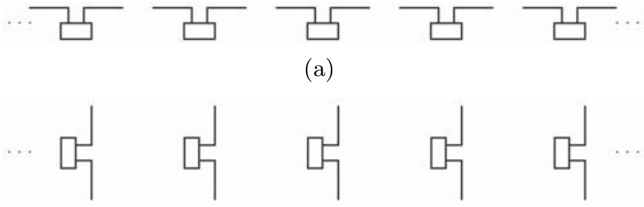
6.2 Electroinductive waves

Magnetoinductive waves may travel on magnetically coupled elements, e.g. on a set of split ring resonators. Beruete et al. realised [10] that by taking the dual of a split ring resonator which exchanges metal for hole, and hole for metal as shown in Figure 3, they can retain the resonant character of the elements but propagate the electric equivalents of magnetoinductive waves. The coupling is now electric described by a mutual impedance Z_m which may have different values in the two media and across the boundary. In order to find the reflection and transmission coefficients, we need to substitute

$$h = Z_m, \quad h' = Z_m' \text{ and } \quad h_b = Z_{mb} \qquad (18)$$

into equations (6).

Fig. 3. Electro-inductive waveguide based on a patterned metal film.



(a)

(b)

Fig. 4. Dipole chains in (a) planar and (b) axial configurations.

6.3 Waves on axial and transverse electric dipole chains

The elements considered are now short dipoles loaded with an impedance at the centre [11] which could tune the elements to the required resonance frequency. They may be arranged axially or side-by-side as may be seen in Figures 4a and 4b. The variable is the electric field across the gap of the dipoles. The coupling between them is due to the longitudinal and transverse electric fields for the axial and side-by-side configurations respectively. For finding the effect of the boundary the same arguments apply as in the previous two sections. The coupling is electric hence the relevant mutual impedances need to be substituted into equations (6).

6.4 Plasma waves on metallic nanoparticle chains

It has been recently shown (see e.g. [12]) that waves on chains of small (relative to the wavelength) metallic spheres may propagate in the form of either longitudinal or transverse electric dipoles. Their resonant character is due to the plasma resonance of the spheres and the coupling between them is dipolar. Both forward and backward waves can be obtained with transverse and longitudinal dipoles respectively. The coefficients g and h are given in reference [12] as

$$g = 1 - \omega^2/\omega_o^2, \quad h = 2r_o^3/d^3,$$
$$h_b = 2r_o^3/d_b^3, \quad \omega_o = \omega_p/3^{1/2} \tag{19}$$

for transverse dipoles, and

$$g = 1 - \omega^2/\omega_o^2, \quad h = 4r_o^3/d^3, \quad h_b = 4r_o^3/d_b^3 \tag{20}$$

for longitudinal dipoles, where r_o is the radius of the metallic sphere, ω_o is the resonance frequency, ω_p is the plasma frequency, and d is the distance between the elements. The spherical array in Medium 2 may be made of a different metal so it may have a different plasma frequency, ω_p' and a different distance between the elements, d'. The distance between the elements on either side of the boundary is d_b. Note that for simplicity r_o is assumed to be the same in both media. The parameters in both media and across the boundary can be substituted once more into equations (6) to obtain the reflection and transmission coefficients.

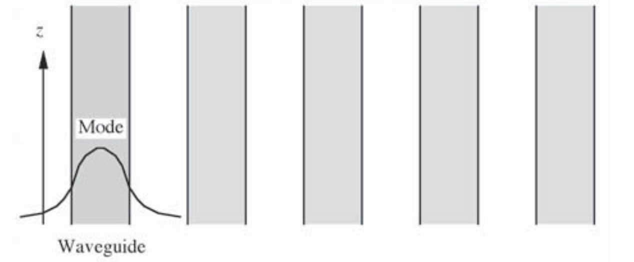


Fig. 5. Co-directionally coupled dielectric waveguide array.

6.5 Coupled waveguide arrays

The original problem [5] that led to the present approach was the guiding of waves by an array of coupled optical waveguides carrying power from a set of mode-locked diode lasers. A schematic representation of the arrangement of waveguides is shown in Figure 5. Assuming nearest neighbour coupling and identical velocities of propagation along the uncoupled waveguides the following, differential equation relating the amplitudes of the modes in adjacent waveguides with distance z along the guide can be derived

$$dA_n/dz = -j\kappa(A_{n-1} + A_{n+1}) \tag{21}$$

where A_n is the amplitude of the mode in the nth guide and κ is the coupling coefficient between neighbouring guides. Assuming propagation along the waveguides in the form

$$A_n = E_n \exp(-j\Delta\beta z) \tag{22}$$

where $\Delta\beta$ is a correction to the propagation coefficient β_0 of the unperturbed guide, equation (21) may be rewritten as

$$\Delta\beta E_n = \kappa(E_{n-1} + E_{n+1}). \tag{23}$$

It may be seen that equation (23) corresponds to the generic form of equation (1). The coupling across the boundary being κ_b we may obtain the reflection and transmission coefficients by substituting

$$h = -\kappa, \quad h' = -\kappa' \text{ and } \quad h_b = -\kappa_b \tag{24}$$

into equations (6). In this case, the results of [5] are obtained.

7 Acoustic waves in solids

None of the examples given in Section 6 cover familiar territory. They are concerned with waves that have received attention only in the recent past and this is the first time

$$R = \frac{hh'\{\exp(-j[kd-(kd)']-1\} - hq'(\exp(-jkd)-1) - h'q(\exp(j(kd)')-1)}{[h_b^2 + \{q - h\exp(jkd)\}\{q' - h'\exp(j(kd)')\}]}$$
$$T = \frac{-2jhh_b\sin(kd)\exp(j(kd)')}{[h_b^2 + \{q - h\exp(jkd)\}\{(q' - h'\exp(j(kd)')\}]}. \quad (29)$$

that their reflection has been discussed. The problem of a boundary between two monatomic acoustic media is however an old subject which has been amply discussed in the literature. We have not included it among our examples in the previous section partly because the problem is somewhat more complicated and partly because in that case we do know the macroscopic equations to which the microscopic equations should reduce.

The transition from the microscopic case to the macroscopic one has had some difficulties in the past. For example, Brillouin [1] made the entirely unphysical assumption that the mass on the boundary is equal to the average mass. We wish to show that our approach leads to the correct macroscopic equations without the need for such an assumption.

The starting point for describing longitudinal acoustic waves in the monatomic case is well known (see e.g. [13]). The interaction between nearest neighbours may be obtained by writing the equation of motion for the nth atom in the chain. It takes the form

$$md^2y_n/dt^2 = -f(y_n - y_{n-1}) - f(y_n - y_{n+1}) \quad (25)$$

where y_n stands for the displacement of the nth atom, m is the mass, t is time, and f is the force constant. With the $\exp(j\omega t)$ time dependence the generic coefficients of equation (1), g and h may be easily obtained from equation (25) as

$$g = -\omega^2 m + 2f \text{ and } \quad h = -f. \quad (26)$$

The same equation appears in Medium 2 but m and f have there the 'dash' superscript. So far there is no deviation from the description in Section 2. However, when we take into account the coupling across the boundary, the resulting equations are slightly more complicated than those of equations (4). They take the form

$$(g+q)x_o + hx_{-1} + h_bx_1 = 0$$
$$(g'+q')x_1 + h_bx_o + h'x_2 = 0. \quad (27)$$

Two new constants q and q' can be seen, which are related to values on the boundary as:

$$q = h - h_b = f_b - f \text{ and } q' = h' - h_b = f_b - f'. \quad (28)$$

From here on we shall use the same approach as before. The reflection and transmission coefficients can now to be obtained from equations (6) in the form

see equation (29) above

For the continuous case when kd, $(kd)' \ll 1$ equations (29) reduce to the well-known forms

$$R = (Z_a' - Z_a)/(Z_a' + Z_a) \text{ and } T = 2Z_a/(Z_a' + Z_a) \quad (30)$$

where

$$Z_a = (fm)^{1/2} \text{ and } Z_a' = (f'm')^{1/2} \quad (31)$$

are the acoustic impedances [13] in Media 1 and 2 respectively. Note that the acoustic impedance plays an entirely analogous role to the wave impedance in electromagnetic boundary reflection phenomena.

8 Further generalisations

The approach used in the present paper can be easily generalised to two-dimensional problems, and this has already been done for magneto-inductive waves in [6]. The technique for finding the amplitudes of the reflected and refracted waves is still the same but it is necessary first to find the angle of refraction. If the 2D dispersion equations are known on both sides of the boundary, then a simple construction yields the angle of the refracted wave provided the incident wave is known.

Generalisation to wave propagation governed by higher order interactions is also straightforward. If, for example, next-nearest neighbour interactions are involved then we need to consider two columns on either side of the boundary and write the recursive equations for four elements, yielding four equations. There are then four unknowns, the reflection and transmission coefficients as before, but in addition there are the unknown amplitudes of two evanescent waves, which will decline away from the boundary.

We would also like to add that the theory is still valid when both electric and magnetic coupling are present, as long as only nearest neighbours are affected. This would happen (for example) in a chain of Split Ring Resonators if the cross-polarization tensor pointed out by Marques et al. [14] were taken into account. The calculation of the coupling coefficient would then be more complicated, but it would still be possible to calculate unique values of h, h' and h_b.

9 Conclusions

The reflection and transmission of a plane wave at the boundary of two periodic media have been investigated. It has been shown that reflection and transmission coefficients can be derived for a generic case valid for a variety of waves whose propagation may be described by nearest-neighbour interaction, e.g. magneto- and electro-inductive waves, waves on short dipoles, on nanoparticles and on coupled waveguides. The boundary problem for acoustic waves in a monatomic medium has also been treated.

It has been shown that the approach is suitable for deriving new microscopic equations for the reflection and transmission coefficients without any artificial assumptions or approximations about the boundary. It has further been shown that the known macroscopic equations in terms of acoustic impedances may be obtained from the microscopic equations, in the continuous limit.

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References

1. L. Brillouin, *Wave propagation in periodic structures: electric filters and crystal lattices*, 2nd edn. (Dover Publications Inc., New York, 1953)
2. R.E. Collin, *Field theory of guided waves* (Oxford University Press, 1991)
3. P.A. Belov, C.R. Simovski, Phys. Rev. B **73**, 045102 (2006)
4. S. Tretyakov, *Analytical modeling in applied electromagnetics* (Artech House, Boston, 2000)
5. R.R.A. Syms, IEEE J. Quant. Electr. **QE-23**, 525 (1987)
6. R.R.A. Syms, E. Shamonina, L. Solymar, Eur. Phys. J. B **46**, 301 (2005)
7. E. Shamonina, V.A. Kalinin, K.H. Ringhofer, L. Solymar, Elect. Lett. **38**, 371 (2002)
8. E. Shamonina, V.A. Kalinin, K.H. Ringhofer, L. Solymar, J. Appl. Phys. **92**, 6252 (2002)
9. L. Solymar, D. Walsh, *Electrical properties of materials*, 7th edn. (Oxford University Press, 2004)
10. M. Beruete, M.J. Freire, R. Marques, F. Falcone, J.D. Baena, Appl. Phys. Lett. **88**, 083503 (2006)
11. S.A. Tretyakov, A.J. Viitanen, El. Engng. **82**, 353 (2000)
12. W.H. Weber, G.W. Ford, Phys. Rev. B **70**, 125429 (2004)
13. J. Dekker, *Solid State Physics* (MacMillan, London, 1952)
14. R. Marqués, F. Medina, R. Rafii-El-Idrissi, Phys. Rev. B **65**, 144440 (2002)